The global Σ^1_{n+1}-Uniformization Property and BPFA

Stefan Hoffelner[*]

April 2, 2024

Abstract

We show that given a reflecting cardinal, one can produce a model of BPFA where the Σ^1_n-uniformization property holds simultaneously for all $n \geq 2$.

1 Introduction

Given $A \subset \omega^\omega \times \omega^\omega$, we say that f is a uniformization (or a uniformizing function) of A if there is a function f such that

$$dom(f) = pr_1(A) = \{x \in 2^\omega \, : \, \exists y((x,y) \in A)\}$$

and the graph of f is a subset of A.

Definition 1.1 (Uniformization Property)**.** *We say that the projective point-class* $\Gamma \in \{\Sigma^1_n \mid n \in \omega\} \cup \{\Pi^1_n \mid n \in \omega\}$ *has the uniformization property iff every Γ-set in the plane admits a uniformization whose graph is in Γ, i.e. the relation $(x,y) \in f$ is in Γ.*

It is a classical result due to M. Kondo that lightface Π^1_1-sets do have the uniformization property, this also yields the uniformization property for Σ^1_2-sets. This is all ZFC can prove about the uniformization property of projective sets. In the constructible universe L, for every $n \geq 3$, Σ^1_n does have the uniformization property which follows from the existence of a good wellorder by an old result of Addison (see [1]). Recall that a Δ^1_n-definable wellorder $<$ of the reals is a *good* Δ^1_n-wellorder if $<$ is of ordertype ω_1 and the relation $<_I \subset (\omega^\omega)^2$ defined via

$$x <_I y \Leftrightarrow \{(x)_n \, : \, n \in \omega\} = \{z \, : \, z < y\}$$

where $(x)_n$ is some fixed recursive partition of x into ω-many reals, is a Δ_n^1-definable. It is easy to check that the canonical wellorder of the reals in L is a good Δ_2^1-wellorder so the Σ_n^1-uniformization property follows for $n \geq 2$. The very same argument also shows that inner models of the form $L[U]$ for U a normal, κ-complete ultrafilter on κ share the same L-like pattern for uniformization.

On the other hand, large cardinal assumptions draw a very different picture. Due to the celebrated result of Moschovakis (see [14] 39.9), PD implies that Π_{2n+1}^1 and Σ_{2n+2}^1-sets have the uniformization property for $n > 1$. By the famous Martin-Steel result (see [15], Theorem 13.6.), the assumption of infinitely many Woodin cardinals outright implies PD, and hence large cardinals fully settle the behaviour of the uniformization property within the projective hierarchy.

The connection of PD with forcing axioms is established via core model induction. Under the assumption of the proper forcing axiom, Schimmerling, Steelt and Woodin showed that PD is true (in fact much more is true, see [17]), thus also under PFA the Π_{2n+1}^1-uniformization holds for $n > 1$. As the uniformization property for one pointclass rules out the uniformization property of the dual pointclass, the behaviour of sets of reals in L and under PFA contradict each other.

It is natural to investigate uniformization in the presence of weaker forcing axioms, the bounded proper forcing axiom BPFA (introduced by Goldstern-Shelah, see [6]) being the paradigmatic example together with Martin's Axiom. A first step was taken in [10], where it is shown that BPFA and "ω_1 is not inaccessible to reals" outright imply that the boldface $\mathbf{\Sigma_3^1}$-uniformization property holds. However Addisons result from 1959 has been the only tool to obtain the L-like global uniformization pattern. As a consequence the known universes which satisfy the Σ_n^1 uniformization property for all $n > 1$ are quite special, in particular they must satisfy CH, hence BPFA fails in such universes.

The main goal of this paper is to produce a new method which will force the L-like uniformization pattern. We will introduce a family of Σ_n^1-predicates whose truth value can be switched from false to true using carefully designed forcings which are ω_1-preserving but not proper. These predicates in turn will be used to define an iterated forcing construction yielding a universe in which the Σ_n^1-uniformization property holds for each $n \geq 2$ simultaneously. The definitions of the uniformizing functions are robust under additional usage of proper forcings, which opens the possibility of forcing BPFA simultaneously.

Theorem 1.2. *Working in L, assuming the existence of a reflecting cardinal, there is a generic extension of L in which* BPFA *and the Σ_n^1-uniformization property for every $n \geq 2$ are true.*

The forcing method which yields the Σ_n^1-uniformization property seems

to be quite flexible and we expect several further applications.

We end the introduction with a short description of how the article is organized. Section 1 introduces the forcings we will use. This builds on a technique first introduced in [12] which in turn relies on the notion of mutual stationarity by Foreman-Magidor and Todorcevic's argument that PFA implies the failure of square. Section 1.3 sets up a coding machinery which can be combined with the arguments for obtaining BPFA without dangerous interferences. Section 2 introduces a family of Σ_n^1-formulas which we will use for the uniformization property. Section 3 explains the coding method which produces universes with the L-like uniformization property, and section 4 combines this method with the argument to obtain BPFA to finally prove the main theorem of this work.

It belongs to a series of articles which devote themselves to the study of the separation, the reduction and the uniformization property (see [8],[9], [10] and [11]). We emphasize however that our arguments in this work (which deal with the Σ-side of uniformization) must differ substantially from the arguments which deal with the Π-side, (as [8],[9] and [11]) as the two pictures necessarily can not be combined within ZFC.

1.1 Mutually stationary sets

An algebra $\mathcal{A}$ on a cardinal λ is a structure of the form $(\lambda, f_i)_{i\in\omega}$, where for each $i \in \omega$, $f_i : [\lambda]^{<\omega} \to \lambda$. Typically, the f_i's will be definable Skolem functions over structures of the form $(H(\theta), \in, <)$ where $<$ is a well-order of $H(\theta)$ and θ is some regular cardinal. The following notion was introduced by M. Foreman and M. Magidor in their seminal [5].

Definition 1.3. *Let K be a collection of regular cardinals whose supremum is strictly below κ, and suppose that $S_\eta \subseteq \eta$ for each $\eta \in K$. Then the collection of sets $\{S_\eta \mid \eta \in K\}$ is* mutually stationary *if and only if for all algebras $\mathcal{A}$ on κ, there is an $N \prec \mathcal{A}$ such that*

$$\text{for all } \eta \in K \cap N, \ \sup(N \cap \eta) \in S_\eta.$$

Foreman-Magidor ([5]) show that every sequence $\vec{S} = (S_\eta \mid \eta \in K)$ of stationary sets which concentrate on ordinals of countable cofinality is mutually stationary. For a fixed sequence $\vec{S}$ of stationary sets, let $\mathcal{T}_{\vec{S}}$ be the collection of all countable N such that for all $\eta_i \in N$, $\sup(N \cap \eta_i) \in S_i$.

Theorem 1.4 (Foreman-Magidor)**.** *Let $(\eta_i \mid i < j)$ be an increasing sequence of regular cardinals. Let $\vec{S} = (S_i \mid i < j)$ be a sequence of stationary sets such that $S_i \subseteq \eta_i \cap Cof(\omega)$. If θ is a regular cardinal greater than all η_i and $\mathcal{A}$ is an algebra on θ, then there is an $N \prec \mathcal{A}$ which belongs to the class $\mathcal{T}_{\vec{S}}$. Hence $\vec{S}$ is mutually stationary.*

From now on, we assume all stationary subsets of ordinals discussed in this section are concentrated on countable cofinality. The corresponding notion for being club in this context is that of an unbounded set which is closed under ω-sequences.

Definition 1.5. *Suppose $\vec{S} = \{S_\eta \mid \eta \in K\}$ is mutually stationary and that for every $\eta \in K$, S_η is stationary, co-stationary in $\eta \cap Cof(\omega)$. We say a forcing poset $\mathbb{P}$ is $\vec{S}$-preserving if the following holds: Suppose $\theta > 2^{|\mathbb{P}^+|}$ is regular. Suppose M is a countable elementary submodel of $H(\theta)$ with $\{\mathbb{P}, \vec{S}\} \subset M$ and $M \in \mathcal{T}_{\vec{S}}$. Suppose $p \in \mathbb{P} \cap M$. Then there exists a $(M, \mathbb{P})$-generic condition q extending p.*

We add some remarks concerning the notions:

1. Any proper forcing is $\vec{S}$-preserving.

2. When $K = \{\omega_1\}$ and $\vec{S} = S \subset \omega_1$, the definition of $\vec{S}$-preserving is identical to the usual definition of S-proper forcing.

3. Let $\vec{S}$ be such that each $S_\eta \in \vec{S}$ is stationary, co-stationary in $\eta \cap Cof(\omega)$. Then an example of a non-proper, $\vec{S}$ preserving forcing is the forcing poset $Club(S_\eta)$ for a fixed η, i.e, the forcing which adds an unbounded subset to S_η which is closed under ω-sequences, via countable approximations.

The preservation theorems for countable support iterations of proper forcings can be generalized to $\vec{S}$-preserving forcings.

Lemma 1.6. *If $\langle P_i, \dot{Q}_i \mid i < \alpha \rangle$ is a countable support iteration of forcing notions and for each $i < \alpha$, $\Vdash_{P_i} "\dot{Q}_i$ is $\vec{S}$-preserving" then P_α is $\vec{S}$-preserving.*

Proof. (Sketch, following the proof of [13], Theorem 31.15, in particular Lemma 31.17) We will only need to show by induction on $j \leq \alpha$ that for any $N \in \mathcal{T}_{\vec{S}}$, if $j, \langle P_i, \dot{Q}_i \mid i < \alpha \rangle \in N$, then:

$(*)_N$ For every $q_0 \in N \cap P_j$ that is (N, P_j)-generic and every $\dot{p} \in V^{P_j}$ such that

$$q_0 \Vdash_j \dot{p} \in (P_\alpha \cap N) \wedge \dot{p} \restriction j \in \dot{G}_j$$

there is an (N, P_α) generic condition $q \in P_\alpha$ extending q_0, that is $q \restriction j = q_0$ and $q \Vdash_\alpha \dot{p} \in \dot{G}_\alpha$.

The statement $(*)_N$ is identical to Lemma 31.17 in [13]. It can be checked that the original proof also works here, which gives the iteration theorem exactly as in the proof of Theorem 31.15. $\square$

We will use club shooting forcings relative to definable sequences $\vec{S}$ of mutually stationary sets to code information. These codings do not interfere with the proper forcings we will use to work towards BPFA.

Definition 1.7. *Let κ be an inaccessible cardinal. Let $\vec{S} = (S_i \mid i < \kappa)$ be mutually stationary. We say a forcing poset $\mathbb{P}$ is an $\vec{S}$-coding if $\delta \leq \kappa$ and $\mathbb{P} = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha < \delta \rangle$ satisfies the following:*

- *$\mathbb{P}$ is a countable support iteration.*

- *For any $\alpha < \delta$, one of the followings holds:*

    1. *Assume that α is inaccessible and $\mathbb{P}_\alpha$ is forcing equivalent to a forcing of size less than or equal to α.[1] Assume that in $V^{\mathbb{P}_\alpha}$, $(B_\beta \subset \alpha \mid \beta < 2^\alpha)$ is an enumeration of an arbitrary set $X \subset P(\alpha)$. Then $\dot{\mathbb{Q}}_\alpha$ is allowed to be the countably supported product $\prod_{\beta < 2^\alpha} \dot{\mathbb{R}}_\beta$, where each $\dot{\mathbb{R}}_\beta$ is itself defined to be*

        $$\prod_{j \in B_\beta} Club(S_{\alpha \cdot (\beta+1)+2j}) \times \prod_{j \notin B_\beta} Club(S_{\alpha \cdot (\beta+1)+2j+1})$$

        using countable support.
    2. *In all other cases, we have that $\Vdash_{\mathbb{P}_\alpha} \dot{\mathbb{Q}}_\alpha$ is proper.*

Let η be an regular cardinal, we say $\mathbb{P}$ is an η-$\vec{S}$ coding if (1) is replaced by

(1') $\alpha \geq \eta$ is inaccessible and $\mathbb{P}_\alpha$ is forcing equivalent to a forcing of size less than or equal to α. In $V^{\mathbb{P}_\alpha}$, $\langle B_\beta \mid \beta < 2^\alpha \rangle$ is an enumeration of $P(\alpha)$. Then $\dot{\mathbb{Q}}_\alpha$ is allowed to be the countably supported product $\prod_{\beta < 2^\alpha} \dot{\mathbb{R}}_\beta$ where each $\dot{\mathbb{R}}_\beta$ is itself

$$\prod_{j \in B_\beta} Club(S_{\alpha \cdot (\beta+1)+2j}) \times \prod_{j \notin B_\beta} Club(S_{\alpha \cdot (\beta+1)+2j+1}).$$

By Lemma 1.6, once we can show that every factor of an $\vec{S}$-coding is $\vec{S}$-preserving, we can deduce that if $\mathbb{P}$ is a $\vec{S}$-coding forcing, then $\mathbb{P}$ is $\vec{S}$-preserving. This assertion follows from the proof of the next lemma which says that we will not accidentally code unwanted information whenever we use a $\vec{S}$-coding forcing.

Lemma 1.8. *Suppose that $\vec{S}$ is stationary, co-stationary. Suppose $\mathbb{P}$ is an $\vec{S}$-coding forcing. Then for any $i \in \kappa$, the followings are equivalent:*

(a) *$\Vdash_{\mathbb{P}} S_i$ contains an ω-club.*

(b) *there are β, α, j and sets $B_\beta \in X \subset V_\alpha$ such that $j \in B_\beta$ if $\beta \cdot (\alpha + 1) + 2j = i$ and i is even and $j \notin B_\beta$ if $\beta \cdot (\alpha + 1) + 2j + 1 = i$ and i is odd.*

---

[1]We say two forcing P and Q are equivalent if their Boolean completions $B(P)$ and $B(Q)$ are isomorphic.

Proof. ((b) → (a)) Follow from the definition of the forcing.

((a) → (b)) Fix an i and assume without loss of generality that i is even. Write $i = \beta \cdot (\alpha + 1) + 2j$ and suppose for a contradiction that j is not an element of B_β. By the definition of $\vec{S}$ coding forcing, we must have added a club through $S_{\beta(\alpha+1)+2j+1}$ instead. Let $\vec{T}$ be the sequence $\langle T_k \mid k < \kappa \rangle$, where $T_k = S_k$ if $k \neq i$ and $T_i = \eta_i \setminus S_i$. It follows from Theorem 1.4 again that $\vec{T}$ is mutually stationary. We will prove that $\mathbb{P}$ is $\vec{T}$-preserving to derive a contradiction. Indeed, we shall see that $\vec{T}$-preservation implies that $\eta_i \setminus S_i$ must remain stationary after forcing with $\mathbb{P}$, yet $\mathbb{P} \Vdash$ "S_i contains an ω-club" which is impossible.

To see that $\vec{T}$ preserving forcings preserve the stationarity of every $S_{\eta_i} \in \vec{T}$, we only need to note that for any name $\dot{C}$ of a subset of η_i which is unbounded and ω-closed, and any countable elementary substructure N which contains $\dot{C}$ and for which $\sup(N \cap \eta_i) \in S_{\eta_i}$, any $(N, \mathbb{P})$-generic condition q forces $\dot{C} \cap (S_{\eta_i}) \neq \emptyset$.

Next we show by induction that each $\dot{Q}_\beta$ is forced to be $\vec{T}$-preserving. Work in $V[G_\beta]$. If $\dot{Q}_\beta / G_\beta$ is proper, then it is also $\vec{T}$-preserving. Otherwise, (1) holds. Now $\dot{Q}_\beta / G_\beta$ is a countable support product of club adding forcings. Fix any $N \in \mathcal{T}_{\vec{T}}$ which is a countable substructure of $H(\theta)^{V[G_\beta]}$. For any $p \in N \cap \dot{Q}_\beta$, we can construct a countable decreasing sequence of conditions $\langle p_n \mid n < \omega \rangle$ meeting all dense set in N. Define q coordinatewise by setting $q(j)$ to be the closure of $\bigcup_{n<\omega} p_n(j)$ if $i \in N$ and trivial otherwise. Note that any non-trivial $q(j)$ is equal to $\bigcup_{n<\omega} p_n(j) \cup \{\sup(N \cap \eta_j)\}$, where $\eta_j = \sup(S_j)$ is a regular cardinal. As $N \in \mathcal{T}_{\vec{T}}$ we have $\sup(N \cap \eta_j) \in S_j$, whenever $q(j)$ is non-trivial. Hence $q < p$ is a condition witnessing that each factor of the iteration is $\vec{T}$-preserving, so the iteration $\mathbb{P}$ is $\vec{T}$-preserving as well. But now S_i must remain stationary after we forced with $\mathbb{P}$, which is a contradiction to (a). □

The proof also shows that $\vec{S}$-coding preserves stationary subset of ω_1 if $\sup(S_0) > \omega_1$. As a Corollary of Lemma 1.8 and the definition of $\vec{S}$-coding, in any generic extension by $\vec{S}$-coding and any even i, at most one of S_i and S_{i+1} contains a club.

The next lemma follows immediately from the definitions so we skip its proof.

Lemma 1.9. *Suppose $\mathbb{P} = \langle P_\alpha, \dot{Q}_\alpha \mid \alpha < \delta \rangle$ is a countable support iteration. Suppose for any $\alpha > 0$, $\dot{Q}_\alpha$ is forced to be η_α-$\vec{S}$ coding of length $l(\alpha)$, where $\eta_\alpha = \max\{|\mathbb{P}_\alpha|^+, \Sigma_{\beta < \alpha} l(\beta)\}$. Also let η_0 be regular. Then $\mathbb{P}$ is forcing equivalent to an η_0-$\vec{S}$ coding.*

1.2 Specializing threads through □-sequences

We will briefly present a celebrated theorem, due to S. Todorcevic, which we use to find Σ_1-definitions of regular L- cardinals. The argument is again entirely due to Todorcevic (see [19]).

Theorem 1.10 (Todorcevic, see [18]). *Let $\kappa > \omega_1$ be a regular cardinal and let $\Gamma \subset \kappa$ be a set of limit ordinals such that $\{\delta < \kappa \mid cf(\delta) = \omega_1\} \subset \Gamma$. Let $(C_\alpha \mid \alpha \in \Gamma\}$ be a sequence of subsets of κ such that*

1. *$\forall \alpha \in \Gamma$ $C_\alpha \subset \alpha$ and C_α is a club,*

2. *if β is a limit point of C_α, then $\beta \in \Gamma$ and $C_\beta = C_\alpha \cap \beta$.*

3. *There is no club $C \subset \kappa$ such that if α is a limit point of of C, then $\alpha \in \Gamma$ and $C_\alpha = C \cap \alpha$.*

Then there is a proper forcing $\mathbb{T}^\kappa$ such that $\mathbb{T}^\kappa$ forces that in the generic extension, there is a closed subset D of κ of ordertype ω_1 and a function $f : D \to \omega$ such that for all α, β, if $\alpha, \beta \in D$ and $\alpha \in \lim C_\beta$, then $f(\alpha) \neq f(\beta)$.

sketch of a proof. For $\alpha, \beta \in \Gamma$ we let $\beta <_T \alpha$ if and only if β is a limit point of C_α. This induces a tree ordering on Γ.

Next we turn to some very useful definitions. For a regular, sufficiently large cardinal λ, an elementary chain $(M_\alpha \prec H(\lambda) \mid \alpha < \omega_1)$ is a sequence of elementary submodels of $H(\lambda)$ which additionally satisfies that $M_\alpha \subset M_\beta$ and $M_\alpha \in M_\beta$ whenever $\alpha < \beta$; and $M_\alpha = \bigcup_{\beta < \alpha} M_\beta$ if α is a limit ordinal.

We now define the desired forcing $\mathbb{T}^\kappa$, which is a prototype for the extremely useful class of forcing with side conditions, as follows: A condition $p \in \mathbb{T}^\kappa$ is a pair $((N_\alpha \mid \alpha \in E), f)$ where

1. E is a finite subset of ω_1 and there is an elementary chain $(M_\alpha \mid \alpha < \omega_1)$ of countable, elementary substructures of $H(\kappa^+)$ such that $N_\alpha = M_\alpha$ for each $\alpha \in E$.

2. f is a specializing function, i.e. if $\delta_{N_\alpha} := \sup(N_\alpha \cap \kappa$ then f is a function from $\{\delta_{N_\alpha} \mid \alpha \in E\}$ into ω such that $f(\gamma) \neq f(\delta)$ whenever $\gamma <_T \delta$.

The key assertion is now that $\mathbb{T}^\kappa$ is proper provided we assume that there is no κ-chain in the tree T on Γ.

Assuming that $\mathbb{T}^\kappa$ is proper, it is standard argument to verify that $\mathbb{P}$ indeed adds an ω_1-closed set D and a specializing $f : D \to \omega$. □

From now on we assume that $0^\#$ does not exist. Let $(C_\alpha \mid \alpha \in \mathrm{Sing}\}$ be Jensen's global □-sequence in L; here Sing denotes the set of limit ordinals α for which $cf(\alpha)^L < \alpha$. Recall that

1. $\forall \alpha \in \mathrm{Sing}(C_\alpha \subset \alpha)$ and C_α is club.

2. $\forall \alpha \in \mathrm{Sing}(\mathrm{o.t.}(C_\alpha) < \alpha)$.

3. $\forall \beta \in \mathrm{Lim}(C_\alpha)(\beta \in \mathrm{Sing} \wedge C_\beta = C_\alpha \cap \beta)$.

We define a tree T on Sing via setting $\alpha <_T \beta$ iff α is a limit point of C_β. Note that for a cardinal λ, the tree for the corresponding $\square_\lambda$-sequence does not have a λ^+-branch, thus $\square_\lambda$ satisfies the three properties from Todorcevic's theorem. As a consequence, the proper poset $\mathbb{T}^{\lambda^+}$ adds a closed set D in $(\lambda^+)^L$ of ordertype ω_1 and simultaneously specializes the tree T restricted on ordinals in D, i.e. the forcing adds a function $f : D \to \omega$ such that for $\alpha, \beta \in D$, if $\alpha <_T \beta$ then $f(\alpha) \neq f(\beta)$.

Consider now the Σ_1-formula $\psi(x, y)$ which says that x is a set of ordinals which are all singular in L, x is closed and of ordertype ω_1; y is a function from x into ω such that $y(\alpha) \neq y(\beta)$ whenever $\alpha, \beta \in x$ and $\alpha \in \lim C_\beta$.

We claim that if $\theta = sup(D)$ and $D \subset \theta' < \omega_2$, and if $f : D \to$ is such that for $\alpha, \beta \in D$, if $\alpha <_T \beta$ then $f(\alpha) \neq f(\beta)$, then θ' is a regular cardinal in L.

Indeed assume θ' were singular in L, then $C_{\theta'}$ would have been defined, so $D \cap \lim C_{\theta'}$ is uncountable. In particular, there is $\alpha < \beta \in D$ such that $f(\alpha) = f(\beta)$, yet $\alpha <_T \beta$, which is a contradiction. To summarize:

Theorem 1.11. *For every L-regular cardinal κ, there is a proper forcing denoted by $\mathbb{T}^\lambda$ and a Σ_1-formula ϕ such that in the generic extension $V[\mathbb{T}^\kappa]$,*

$$V[\mathbb{T}^\kappa] \models \phi(\kappa) \Leftrightarrow \kappa \text{ is a regular cardinal in } L$$

1.3 Coding machinery

When applied over L, the $\vec{S}$-coding forcings can be used to surgically alter the truth-value of certain projective formulas. These formulas and the $\vec{S}$-coding forcings will be used to form projective predicates $\Phi_{n,m}$, for $n, m \in \omega$, where each $\Phi_{n,m}$ is a Σ_n^1-formula which will eventually serve as our graph of our uniformizing functions for $A_{n,m}$, where $A_{n,m}$ is the m-th Σ_n^1-set in the plane.

We first describe how to use the $\vec{S}$-coding forcings to obtain suitable Σ_n^1-predicates, whose truth-value can be manipulated using the right forcings.

We work over L as our ground model. Let $r \in L$ be a real. Note that over L there is a sequence

$$\vec{S} = (S_\alpha \mid \alpha \in \mathrm{Lim})$$

uniformly definable satisfying

- $S_\alpha \subset \alpha$

- If α is a regular cardinal, then S_α is stationary co-stationary in $\alpha \cap \text{Cof}(\omega)$.

The existence of such a sequence $(S_\alpha \mid \alpha \in \text{Lim}(\delta))$ follows from the fact that $\diamondsuit_\lambda$ holds in L for any L-cardinal λ and is a routine construction.

We aim to find first a Σ_3^1-formula Φ such that $L \models \neg \exists x \Phi(x)$, yet for $\eta \in \text{Ord}$ there is a coding forcing $\text{Code}(r, \eta) \in L$ such that after forcing with $\text{Code}(r, \eta)$, in the resulting universe $L[\text{Code}(r, \eta)] \models \Phi(r)$ and $\forall s(s \neq r \rightarrow \neg \Phi(s))$ does hold.

The desired forcing $\text{Code}(r, \eta)$ will be itself a three step iteration denoted by $\mathbb{Q}^0(r, \eta) * \dot{\mathbb{Q}}^1(r, \eta) * \dot{\mathbb{Q}}^2(r, \eta)$, where the first and second factor are iterations themselves, and we will describe it now. Given our real r and $\eta \in L$ such that η is a limit cardinal in L, we first look at the ω-block of L-cardinals which follow η, that is we form the interval $[\eta, (\eta^{+\omega})^L)$.

The first factor of the iteration, denoted by $\mathbb{Q}^0(r, \eta)$, will use a fully supported ω-length iteration, with $\mathbb{T}^{\eta^{+i}}$ for $i \in \omega$ as factors. Consequentially, after forcing with this partial order, each L-cardinal in the interval $[\eta, \eta^{+\omega})$ is Σ_1-definable using ω_1 as the only parameter in its definition.

For the second factor of our coding forcing we first identify the real r with the according subset of ω and form the second factor of $\text{Code}(r, \eta)$ as follows:
$$\mathbb{Q}^1(r, \eta) := \prod_{j \in r} Club(S_{\eta+2j}) \times \prod_{j \notin r} Club(S_{\eta+2j+1})$$

using countable support.

After the two forcings are done we define (in the resulting generic extension) $X \subset \omega_1$ to be the $<$-least set (in some previously fixed well-order of some sufficiently large $H(\theta)$) which codes the following objects:

- The $<$-least set of closed sets of η^{+i} of ordertype ω_1 and the according specializing functions, both added with the factors of the forcing of the form $\mathbb{T}^{\eta^{+i}}$. These sets together ensure the Σ_1-definability of the L-cardinals in the interval $[\eta, \eta^{+\omega})$.

- And ω-many ω-closed, unbounded subsets through some of the canonically definable stationary sets $S_{\eta^{+i}} \subset \eta^{+i} \cap \text{cof}(\omega)$. These ω-closed sets are so chosen that the characteristic function of r can be read off. That is we collect $\{c_{\eta^{+i}} \subset S_{\eta^{+i}} : i = 2n \wedge n \notin r\}$ and $\{c_{\eta^{+i}} \subset S_{\eta^{+i}} : i = 2n + 1 \wedge n \in r\}$.

As mentioned already, when working in $L[X]$ then we can read off r via looking at the ω-block of L-cardinals starting at η and determine which canonical L-stationary set $S_{\eta^{+i}}$ contains an ω-closed, unbounded set in $L[X]$:

$(*)$ $n \in r$ if and only if $S_{\eta^{+(2n+1)}}$ contains an ω-closed unbounded set, and $n \notin r$ if and only if $S_{\eta^{+(2n)}}$ contains an ω-closed unbounded set.

Indeed this follows readily from Lemma 1.8.

We note that we can apply an argument resembling David's trick in this situation. We rewrite the information of $X \subset \omega_1$ as a subset $Y \subset \omega_1$ using the following line of reasoning. It is clear that any transitive, $\aleph_1$-sized model M of ZF^- which contains X will be able to correctly decode out of X all the information. Consequentially, if we code the model $(M, \in)$ which contains X as a set $X_M \subset \omega_1$, then for any uncountable β such that $L_\beta[X_M] \models \mathsf{ZF}^-$ and $X_M \in L_\beta[X_M]$:

$$L_\beta[X_M] \models \text{``The model decoded out of } X_M \text{ satisfies } (*)\text{''}.$$

In particular there will be an $\aleph_1$-sized ordinal β as above and we can fix a club $C \subset \omega_1$ and a sequence $(M_\alpha \ : \ \alpha \in C)$ of countable elementary submodels of $L_\beta[X_M]$ such that

$$\forall \alpha \in C(M_\alpha \prec L_\beta[X_M] \wedge M_\alpha \cap \omega_1 = \alpha)$$

Now let the set $Y \subset \omega_1$ code the pair (C, X_M) such that the odd entries of Y should code X_M and if $Y_0 := E(Y)$ where the latter is the set of even entries of Y and $\{c_\alpha \ : \ \alpha < \omega_1\}$ is the enumeration of C then

1. $E(Y) \cap \omega$ codes a well-ordering of type c_0.

2. $E(Y) \cap [\omega, c_0) = \emptyset$.

3. For all β, $E(Y) \cap [c_\beta, c_\beta + \omega)$ codes a well-ordering of type $c_{\beta+1}$.

4. For all β, $E(Y) \cap [c_\beta + \omega, c_{\beta+1}) = \emptyset$.

We obtain

$(**)$ For any countable transitive model M of ZF^- such that $\omega_1^M = (\omega_1^L)^M$ and $Y \cap \omega_1^M \in M$, M can construct its version of the universe $L[Y \cap \omega_1^N]$, and the latter will see that there is an $\aleph_1^M$-sized transitive model $N \in L[Y \cap \omega_1^N]$ which models $(*)$ for r and η.

Thus we have a local version of the property $(*)$.

We know define the third, and last factor of $\mathrm{Code}(r, \eta)$, working in $L[\mathbb{Q}^0(r, \eta)][\dot{\mathbb{Q}}^1(r, \eta)]$ we shall define $\mathbb{Q}^2(r, \eta)$ as follows. We use almost disjoint forcing $\mathbb{A}_D(Y)$ relative to our previously defined, almost disjoint family of reals $D \in L$ (see the paragraph after Definition 2.5) to code the set $Y \subset \omega_1$ into one real r. This forcing only depends on the subset of ω_1 we code, thus $\mathbb{A}_D(Y)$ will be independent of the surrounding universe in which we define it, as long as it has the right ω_1 and contains the set Y.

We finally obtained a real R such that

$(***)$ For any countable, transitive model M of ZF^- such that $\omega_1^M = (\omega_1^L)^M$ and $R \in M$, M can construct its version of $L[R]$ which in turn thinks that there is a transitive ZF^--model N of size $\aleph_1^M$ such that N believes $(*)$ for r and η.

Note that $(\ast\ast\ast)$ is a Π^1_2-formula in the parameters R and r. We will often suppress the sets r, η when referring to $(\ast\ast\ast)$ as they will be clear from the context. We say in the above situation that the real r *is written into* $\vec{S}$, or that r *is coded into* $\vec{S}$ (at η) and R witnesses that r is coded (at η).

The projective and local statement $(\ast\ast\ast)$, if true, will determine how certain inner models of the surrounding universe will look like with respect to branches through $\vec{S}$. That is to say, if we assume that $(\ast\ast\ast)$ holds for a real r and is the truth of it is witnessed by a real R. Then R also witnesses the truth of $(\ast\ast\ast)$ for any transitive ZF^--model M which contains R (i.e. we can drop the assumption on the countability of M). Indeed if we assume that there would be an uncountable, transitive M, $R \in M$, which witnesses that $(\ast\ast\ast)$ is false. Then by Löwenheim-Skolem, there would be a countable $N \prec M$, $R \in N$ which we can transitively collapse to obtain the transitive $\bar{N}$. But $\bar{N}$ would witness that $(\ast\ast\ast)$ is not true for every countable, transitive model, which is a contradiction.

Consequentially, the real R carries enough information that the universe $L[R]$ will see that certain L-stationary sets from $\vec{S}$ have clubs in that

$$n \in r \Rightarrow L[R] \models \text{``}S_{\eta^+(2n+1)} \text{ contains an } \omega\text{-closed unbounded set''}.$$

and

$$n \notin r \Rightarrow L[R] \models \text{``}S_{\eta^+(2n)} \text{ contains an } \omega\text{-closed unbounded set''}.$$

Indeed, the universe $L[R]$ will see that there is a transitive ZF^--model N which believes $(\ast)$. The latter being coded into R. But by upwards Σ_1-absoluteness, and the fact that N can compute $\vec{S}$ correctly, if N thinks that some L-stationary set in $\vec{S}$ contains an ω-closed, unbounded set, then $L[R]$ must think so as well.

2   Suitable Σ^1_n-predicates

We shall use the Σ^1_3-predicate "being coded into $\vec{S}$" (we will often write just "being coded" for the latter) to form suitable Σ^1_n-predicates Φ^n for every $n \in \omega$. These predicates share the following properties:

1. $L \models \forall x \neg(\Phi^n(x))$

2. For every real $x \in L$, there is a coding forcing $\mathrm{Code}^n(x) \in L$ such that after forcing with it, $L[\mathrm{Code}^n(x)] \models \Phi^n(x)$, and for every real $y \neq x$, $L[\mathrm{Code}^n(x)] \models \neg\Phi^n(y)$.

Most importantly, these properties remain true even when iterating the coding forcings $\mathrm{Code}^n(x_i)$ for a sequence of (names of) reals.

The predicates $\Phi^n(x)$ will be defined now.

- $\Phi^3(x, y, m) \equiv \exists a_0((x, y, m, a_0) \text{ is coded into } \vec{S})$.

- $\Phi^4(x, y, m) \equiv \exists a_0 \forall a_1((x, y, m, a_0, a_1) \text{ is not coded into } \vec{S})$.

- $\Phi^5(x, y, m) \equiv \exists a_0 \forall a_1 \exists a_2((x, y, m, a_0, a_1, a_2) \text{ is coded into } \vec{S})$.

- $\Phi^6(x, y, m) \equiv \exists a_0 \forall a_1 \exists a_2 a_3((x, y, m, a_0, a_1, a_2, a_3) \text{ is not coded into } \vec{S})$.

- ...

- ...

- $\Phi^{2n}(x, y, m) \equiv \exists a_0 \forall a_1 ... \forall a_{2n}((x, y, m, a_0, ..., a_{2n}) \text{ is not coded into } \vec{S})$.

- $\Phi^{2n+1}(x, y, m) \equiv \exists a_0 \forall a_1 ... \exists a_{2n+1}((x, y, m, a_0, ..., a_{2n+1}) \text{ is coded into } \vec{S})$.

- ...

- ...

Each predicate Φ^n is exactly Σ_n^1. In the choice of our Σ_n^1-formulas $\Phi^n(x)$, we encounter again a periodicity phenomenon, that is two different cases depending on $n \in \omega$ being even or odd, a theme which is pervasive in this area. It is clear that for each predicate Φ^n and each given real x there is a way to create a universe in which $\Phi^n(x)$ becomes true using our coding forcings. We just need to iterate the relevant coding forcings using countable support.

Lemma 2.1. *Let $n \in \omega$ and let x be a real in our ground model L. Then there is a forcing $\text{Code}^n(x)$ such that if $G \subset \text{Code}^n(x)$ is generic, the generic extension $L[G]$ will satisfy $\Phi^n(x)$ and for every $y \neq x$, $L[G] \models \neg\Phi^n(y)$. This property can be iterated, that is it remains true if we replace L with $L[G]$ in the above.*

3   Based forcing

This section introduces a suitable collection of forcings, dubbed based forcings, which we use for the proof. As we proceed in our iteration, we simultaneously refine based forcings, via adding more and more constraints, yielding a decreasing hierarchy of α-based forcings.

As a first step, we shall consider the problem of forcing the Σ_3^1-uniformization property over universes which satisfy the anti-large cardinal hypothesis that $\omega_1 = \omega_1^L$ (or more generally satisfy that ω_1 is not inaccessible to reals). Recall our Σ_3^1-predicate $\Phi^3(x)$ defined in the last two sections, and the attached coding forcing $\text{Code}(x)$. There is a very easy strategy to force the Σ_3^1-uniformization property for universes V which satisfy $\omega_1 = \omega_1^L$ provided

V thinks that $\Phi^3(x)$ does not hold for every real $x \in V$: we pick some book-keeping function $F \in V$ which should list all the (names of) reals in our iteration and at each stage β, under the assumption that we created already $V[G_\beta]$, if $F(\beta)$ lists a real $x \in V[G_\beta]$ and a natural number k, we ask whether there is a real y such that

$$V[G_\beta] \models \varphi_k(x, y)$$

holds, where φ_k is the k-th Σ_3^1-formula. If so, we pick the least such y (in some fixed well-order), and let the value of the desired Σ_3^1-uniformizing $f_k(x)$ to be y. Additionally we force with $\mathrm{Code}(x, y, k)$ over $V[G_\beta]$ and obtain $V[G_{\beta+1}]$. The resulting universe $V[G_{\beta+1}]$ will satisfy that $\Phi^3(x, y, k)$ is true, whereas $\Phi^3(x, y', k)$ is not true for each $y' \neq y$. Moreover, because of upwards absoluteness of Σ_3^1-formulas, this property will remain true in all further generic extensions we create.

If we iterate long enough in order to catch our tail, the final model $V[G]$ will satisfy the Σ_3^1-uniformization property via

$$f_k(x) = y \Leftrightarrow \Phi^3(x, y, k).$$

To summarize, the easy strategy to force Σ_3^1-uniformization is to consider at each step some x-section of some Σ_3^1-set $A_k \subset \omega^\omega \times \omega^\omega$, and if non-empty, pick the least y for which $A_k(x, y)$ is true. Then force to make $\Phi^3(x, y, k)$ true and repeat.

Based forcings use this strategy for Σ_3^1-sets, while putting no constraints on Σ_n^1-sets for $n > 3$. Anticipating that we aim for a universe of BPFA, we let κ be the least reflecting cardinal in L.

Definition 3.1. *Let $\lambda < \kappa$ and let $F : \lambda \to H(\kappa)$ be a bookkeeping function. We say that an iteration $(\mathbb{P}_\beta, \dot{\mathbb{Q}}_\beta \mid \beta < \lambda)$ is (0-)based with respect to F if the iteration is defined inductively via the following rules: Assume that $F(\beta) = (\dot{x}, 3, \dot{k})$ where $\dot{x}$ is a $\mathbb{P}_\beta$-name of a real and $\dot{k}$ a $\mathbb{P}_\beta$-name of a natural number. Also assume that $V[G_\beta] \models \exists y(\varphi_{3,k}(x, y))$ and let $\dot{y}$ be the $<$-least name of such a real in some fixed well-order of $H(\kappa)$. Then let $\dot{\mathbb{Q}}_\kappa^{G_\beta}$ be $\mathrm{Code}(x, y, 3, k)$. If $V[G_\beta] \models \exists y(\phi_{3,k}(x, y))$ is not true use the trivial forcing.*

Otherwise, we let the bookkeeping decide whether it provides us with a some reals $y, a_0, a_1, ..., a_n$ and whether a tuple of the form $(x, y, a_0, ..., a_n, m, k)$ is coded or not.

3.1 Strategy to obtain global Σ-uniformization

We shall describe the underlying idea to force the global Σ-uniformization. The definition of the factors of the iteration will depend on whether the formula φ_m we consider at our current stage is in Σ_{2n}^1 or in Σ_{2n+1}^1 where $n \geq 2$. We start with the case where φ_m appears first on an odd level of the projective hierarchy. Assume that $\varphi_m \equiv \exists a_0 \forall a_1 \exists a_2 ... \exists a_{2n-2} \psi(x, y, a_0, a_1, ... a_{2n})$ is

a Σ_{2n+1}^1-formula (where $\psi(x, y, a_0, ...a_{2n})$ is a Π_2^1-formula quantifying over the two remaining variables a_{2n-1} and a_{2n}) and x is a real. We want to find a value for the uniformization function for φ_m at the x-section.

To start, we list all triples of reals $((x, y^0, a_0^0), (x, y^1, a_0^1), (x, y^2, a_0^2), ...)$ according to some fixed well-order $<$. There will be a $<$-least triple $(x, y^\alpha, a_0^\alpha)$ for which $\forall a_1 \exists a_2 ... \exists a_{2n-2} \psi(x, y^\alpha, a_0^\alpha, a_1, a_2...)$ is true, otherwise the x-section would be empty and there is nothing to uniformize.

The goal will be to set up the iteration in such a way that all triples (x, y^β, a_0^β), $\beta > \alpha$ will satisfy the following formula, which is Π_{2n+1}^1 in the parameters $(x, y^\beta, a_0^\beta, m)$ as is readily checked:

$\quad \forall a_1 \exists a_2 \forall a_3 ... \exists a_{2n-2}((x, y^\beta, m, a_0^\beta, a_1, a_2..., a_{2n-2})$ is not coded into $\vec{S}$).

At the same time the definition of the iteration will ensure that $(x, y^\alpha, a_0^\alpha)$ will satisfy

$\quad \exists a_1 \forall a_2 \exists a_3 ... \forall a_{2n-2}((x, y^\alpha, m, a_0^\alpha, a_1, a_2..., a_{2n-2})$ is coded into $\vec{S}$).

Provided we succeed, the pair (x, y^α) will then be the unique solution to the following formula, which is Σ_{2n+1}^1, and which shall be the defining formula for our uniformizing function:

$$\sigma_{\text{odd}}(x, y, m) \equiv \exists a_0 (\forall a_1 \exists a_2 ... \psi(x, y, a_0, a_1, ...) \wedge$$
$$\neg(\forall a_1 \exists a_2 ... ((x, y, m, a_0, a_1, a_2...) \text{ is not coded into } \vec{S}))$$

Indeed, for all $\beta > \alpha$, (x, y^β, a_0^β) can not satisfy the second subformula of Ψ whereas for all $\beta < \alpha$, $(x, y^\beta, a_0^\beta$ can not satisfy the first subformula, as $(x, y^\alpha, a_0^\alpha)$ is the least such triple.

If we assume that φ_m is on an even level of the projective hierarchy we will define things in a dual way to the odd case. Assume that $\varphi_m \equiv \exists a_0 \forall a_1 \exists a_2 ... \forall a_{2n-3} \psi(x, y, a_0, a_1, ...a_{2n-1})$ is a Σ_{2n}^1-formula and x is a real. We want to find a value for the uniformization function for φ_m at the x-section.

Again, we list all triples of reals $((x, y^0, a_0^0), (x, y^1, a_0^1), (x, y^2, a_0^2), ...)$ according to some fixed well-order $<$. There will be a $<$-least triple $(x, y^\alpha, a_0^\alpha)$ for which $\forall a_1 \exists a_2 ... \forall a_{2n-3} \psi(x, y^\alpha, a_0^\alpha, a_1, a_2...)$ is true, otherwise the x-section would be empty and there is nothing to uniformize.

The goal will be to set up the iteration in such a way that all triples (x, y^β, a_0^β), $\beta > \alpha$ will satisfy the following formula, which is Π_{2n}^1 in the parameters $(x, y^\beta, a_0^\beta, m)$ as is readily checked:

$\quad \forall a_1 \exists a_2 \forall a_3 ... \exists a_{2n-3}((x, y^\beta, m, a_0^\beta, a_1, a_2..., a_{2n-3})$ is coded into $\vec{S}$).

At the same time the definition of the iteration will ensure that $(x, y^\alpha, a_0^\alpha)$ will satisfy

$\quad \exists a_1 \forall a_2 \exists a_3 ... \exists a_{2n-3}((x, y^\alpha, m, a_0^\alpha, a_1, a_2..., a_{2n-3})$ is not coded into $\vec{S}$).

Provided we succeed, the pair (x, y^α) will then be the unique solution to the following formula, which is Σ^1_{2n}, and which shall be the defining formula for our uniformizing function:

$$\sigma_{\text{even}}(x, y, m) \equiv \exists a_0 (\forall a_1 \exists a_2 ... \psi(x, y, a_0, a_1, ...) \wedge$$
$$\neg (\forall a_1 \exists a_2 ... ((x, y, m, a_0, a_1, a_2 ...) \text{ is coded into } \vec{S}))$$

3.2 Definition of the iteration according to the strategy.

We shall elaborate on the ideas presented in the last section. Our goal is to define an iteration such that our formulas σ_{even} and σ_{odd} work as defining formulas for the uniformization functions. Later we will combine these ideas with the standard iteration to obtain BPFA from a reflecting cardinal to finally finish the proof of the main theorem.

We will need the following predicates. For an arbitrary real x, list the triples $(x, y^0, a_0^0), (x, y^1, a_0^1), (x, y^2, a_0^2), ...$ according to our fixed well-order $<$. For each ordinal $\alpha < 2^{\aleph_0}$ we fix bijections $\pi_\alpha : (2^{\aleph_0})^\alpha \to 2^{\aleph_0}$ (we assume w.l.o.g that such a bijection exist as we always force its existence with a proper forcing).

We say
$$\Psi(\#\psi, x, y^\alpha, a_0^\alpha, b_1, ..., b_n) \text{ is true}$$

if for every $\beta < \alpha$, if we let $\pi_\alpha^{-1}(b^k) = (c_\beta^k)_{\beta < \alpha}$ then $\psi(x, y^\beta, a_0^\beta, c_\beta^1, c_\beta^2, ..., c_\beta^n)$ is true, where ψ is a Π^1_2-formula. Note that strictly speaking, Ψ is not first order as it is a conjunction of α-many formulas. We will use coding to create a projective predicate which is true whenever Ψ is true.

Let $F : \kappa \to H(\lambda)$ be some bookkeeping function in L which shall be our ground model and which guides our iteration. We assume that we have defined already the following list of notions:

- We do have the notion of α_β-based forcings, which is a subset of the set of based forcings. There is a set of tuples of reals which is tied to the α_β-based forcings and which consists of tuples of reals we must not use in coding forcings anymore as factors of an α_β-based forcing.

- We defined already our iteration $\mathbb{P}_\beta \in L$ up to stage β.

- We picked a $\mathbb{P}_\beta$-generic filter G_β for $\mathbb{P}_\beta$ and work, as usual, over $L[G_\beta]$.

The bookkeeping function F at β hands us an ω-tuple $(\dot{x}, \dot{m}, \dot{l}, \dot{b}_0, \dot{b}_1, \dot{b}_2, ...)$ where $\dot{x}$ and $\dot{b}_n$ are $\mathbb{P}_\beta$-names of a reals and $\dot{m}$ and $\dot{l}$ are both $\mathbb{P}_\beta$-names of natural numbers (in fact at each stage we will only need finitely many of those names of reals, and the rest of the information will be discarded). We let $x = \dot{x}^{G_\beta}$, and define b_n, m, l accordingly. Our goal is to define the forcing $\dot{\mathbb{Q}}_\beta$ we want to use at stage β, and to define the notion of $\alpha_\beta + 1$-allowable forcing. We consider various cases for our definition.

3.2.1 Case 1

In the first case, writing $\varphi_m = \exists a_0 \forall a_1 ... \exists a_{2n-2} \psi(x, y, a_0, ..., a_{2n-2})$, where ψ is a Π_2^1-formula (and φ_m is a Σ_{2n-1}^1 formula for $n \geq 3$) we assume that the bookkeeping F at β hands us a tuple $c_1, ... c_{2n-2}$ of real numbers, x, and $\alpha < 2^{\aleph_0}$ with the assigned reals a_0^α and y^α.

Recall that we have our fixed bijection $\pi_\alpha : (2^{\aleph_0})^\alpha \to 2^{\aleph_0}$ (note that such a bijection exists at least in a proper generic extension, so we pass to that universe if necessary) We assume that

$$\psi(x, y, a_0, c_1, ..., c_{2n-2}) \text{ is true.}$$

We first define $\alpha_\beta + 1$-based forcings as just α_β-based with the additional constraints that we must not use any forcings of the form

$$\text{Code}(\#\psi, x, y^i, a_0^i, e_1, e_2, ..., e_{2n-2})$$

as a factor of our iteration, where $i > \alpha$ and $e_1, e_2, ..., e_{2n-2}$ are such that there are reals $r_0^j, r_1^j, ..., r_i^j$ with $r_\alpha^j = c_j$ and $\pi_i(r_0^j, ..., r_i^j) = e_j$ for every $j \in [1, 2n-2]$.

We use the coding forcing

$$\text{Code}(\#\psi, x, y^\alpha, a_0^\alpha, g_1, g_2, ..., g_{2n-2}) =: \dot{\mathbb{Q}}_\beta$$

as the β-th factor of our iteration, where $g_1, ..., g_{2n-2}$ are reals which we are given by the bookkeeping and such that $\Psi(\#\psi, x, y^\alpha, a_0^\alpha, g_1, ..., g_{2n-2})$ is true.

If $\Psi(\#\psi, x, y^\alpha, a_0^\alpha, g_1, ..., g_{2n-2})$ is not true, then pick the $<$-least tuple $g_1, ..., g_{2n-2}$) such that $\Psi(\#\psi, x, y^\alpha, a_0^\alpha, g_1, ..., g_{2n-2})$ is true. If there is no such tuple then do nothing.

The upshot of this definition is the following. Suppose we iterate following our rules, and in the final model $L[G_\kappa]$, $\exists a_0 \forall a_1 ... \exists a_{2n-2} \psi(x, y, a_0, a_1, ..., a_{2n-2})$ is true and (x, y, a_0) is the $<$-least triple witnessing this. Then also

$$\exists a_0 \forall a_1 ... \exists a_{2n-2}((\#\psi, x, y, a_0, a_1, ..., a_{2n-2}) \text{ is coded})$$

is true in $L[G_\kappa]$ by the rules of the iteration. At the same time for every $(x, y'', a_0'') > (x, y, a_0)$ and for every real c_1 there will be a real b_2' such that for every c_3 there is a further b_4' and so on such that $(x, y'', a_0'', \#\psi, c_1, b_2', ...)$ is not coded.

3.2.2 Case 2

We again let $\varphi_m = \exists a_0 \forall a_1 ... \exists a_{2n-2} \psi(x, y, a_0, ..., a_{2n-2})$ and we assume that the bookkeeping F at β hands us a tuple $c_1, ... c_{2n-2}$ of real numbers, x, and $\alpha < 2^{\aleph_0}$ with the assigned reals a_0^α and y^α.

We assume that

$$\psi(x, y, a_0, c_1, ..., c_{2n-2}) \text{ is not true.}$$

In this situation we force with the trivial forcing at the β-th stage and do not define the notion of $\alpha_\beta + 1$-based forcing. Instead we continue working with α_β-based forcings.

3.2.3 Case 3

This is the dual case to the first one. We assume that

$$\varphi_m \equiv \exists a_0 \forall a_1, ..., \forall a_{2n}(\psi(x, y, a_0, ..., a_{2n-3}))$$

that is φ_m belongs to an even projective level. We assume that the book-keeping F at β hands us a tuple $c_1, ...c_{2n-3}$ of real numbers, x, and $\alpha < 2^{\aleph_0}$ with the assigned reals a_0^α and y^α.

We assume that

$$\psi(x, y^\alpha, a_0^\alpha, c_1, ..., c_{2n-3}) \text{ is true.}$$

First we define the β-the factor of our iteration. We force with

$$\text{Code}(\#\psi, x, y^i, a_0^i, e_1, e_2, ..., e_{2n-3}) =: \dot{\mathbb{Q}}_\beta$$

where $i > \alpha$ and $e_1, e_2, ..., e_{2n-3}$ are reals given by the bookkeeping which have the property that there are reals $r_0^j, r_1^j, ..., r_i^j$ with $r_\alpha^j = c_j$ and $\pi_i(r_0^j, ..., r_i^j) = e_j$ for every $j \in [1, 2n - 3]$.

Next we define the notion of $\alpha_\beta + 1$-based which should be α_β-based and the additional constraint that

$$\text{Code}(\#\psi, x, y, \alpha, g_1, g_2, ..., g_{2n-3}) =: \dot{\mathbb{Q}}_\beta$$

must not be used as a factor of any future forcing we use in our iteration. Here $g_1, ..., g_{2n-3}$ are reals which we are given by the bookkeeping with the property that $\Psi(\#\psi, x, y^\alpha, a_0^\alpha, g_1, ..., g_{2n-3})$ (see at the beginning of section 2.2 for a definition of Ψ) holds true.

3.2.4 Case 4

This is the same as in case 2. We find ourselves in the situation where $\psi(x, y^\alpha, a_0^\alpha, c_1, ..., c_{2n-3})$ is not true. Then we do nothing.

This ends the definition of our iteration. We shall argue later, that the definition works in that it will produce a universe where σ_{even} and σ_{odd} serve as definitions of uniformizing functions, when applied in the right context.

Definition 3.2. *Let $F : \kappa \to H(\lambda)$ be a bookkeeping function. We say that an iteration $(\mathbb{P}_\beta, \dot{\mathbb{Q}}_\beta \mid \beta < \kappa)$ is suitable if each factor of the iteration either is proper or each factor is obtained using F and the four rules described above.*

17

4 Proving the main theorem

Recall first that κ is reflecting whenever it is regular and such that V_κ is Σ_2-elementary in the universe V. We let κ be the least reflecting cardinal in L. We let $F : \kappa \to H(\kappa)$ be such that every element of $H(\kappa)$ has an unbounded pre-image under F. We assume that we arrived at stage $\beta < \kappa$ of our iteration and will describe what to force with now.

4.1 Definition of the iteration. Odd stages

Assume that $F(\beta)$ hands us names such that they evaluate with the help of our current generic G_β to a tuple $(\#\psi, x, y^\alpha, a_0^\alpha, c_1, ..., c_n, ...,)$ where $\#\psi$ is the the Gödelnumber of a Σ_n^1-formula and $x, y^\alpha, a_0^\alpha, c_1, ...)$ are reals.

In this situation we just follow the one of the four cases in the definition of suitable forcings which applies.

4.2 Definition of the iteration. Even stages.

We work towards BPFA on the even stages of the iteration. Recall that due to J. Bagaria (see [2]) BPFA is equivalent to the assertion that $H(\omega_2) \prec_{\Sigma_1} V^{\mathbb{P}}$ for any proper forcing $\mathbb{P}$. We shall work towards Bagaria's reformulation of BPFA adapting the usual way.

Assume that the bookkeeping F at stage β hands us a Σ_1-formula σ and parameters $p_0, ..., p_l$ of σ which are elements of $P(\omega_1) \cap L[G_\beta]$. We ask of $\sigma(p_0, ..., p_l)$ whether it holds in a suitable generic extension of $L[G_\beta]$. If yes, then as κ is reflecting, there is such a forcing already in $L_\kappa[G_\beta]$. Moreover the witness to the true Σ_1-formula can be assumed to have a name in $L_\kappa[G_\beta]$. We fix a $\gamma < \kappa$ such that L_γ is Σ_2-elementary in L_κ, which implies that also $L_\gamma[G_\beta]$ is Σ_2-elementary in $L_\kappa[G_\beta]$. Hence the suitable forcing $\mathbb{Q}$ which forces $\sigma(p_0, ..., p_l)$ to become true can be assumed to belong to $L_\gamma[G_\beta]$ already. We force with $\mathbb{Q}$ at stage β then.

4.3 Properties of the defined universe

Lemma 4.1. *Let G_κ denote a generic filter for the full, κ-length iteration.*

1. *Then $L[G_\kappa]$ satisfies that whenever $\varphi_m = \exists a_0 \forall a_1 ... \exists a_{2n-2} \psi(x, y, a_0, ..., a_{2n-2})$ and $(x, y^\alpha, a_0^\alpha)$ is such that*

$$L[G_\kappa] \models \varphi_m(x, y^\alpha, a_0^\alpha)$$

Then for each $\beta > \alpha$

$$L[G_\kappa] \models \forall a_1 \exists a_2 ... \exists a_{2n-2}((\#\psi, x, y^\beta, a_0^\beta, a_1, .., a_{2n-2}) \text{ is not coded })$$

2. If $\varphi_m = \exists a_0 \forall a_1 ... \exists a_{2n-3} \psi(x, y, a_0, ..., a_{2n-3})$ and $(x, y^\alpha, a_0^\alpha)$ is such that

$$L[G_\kappa] \models \varphi_m(x, y^\alpha, a_0^\alpha)$$

Then for each $\beta > \alpha$

$$L[G_\kappa] \models \forall a_1 \exists a_2 ... \exists a_{2n-3}((\#\psi, x, y^\beta, a_0^\beta, a_1, .., a_{2n-3}) \text{ is coded })$$

Proof. We will only proof the first item, as the second is shown in the dual way. Let a_1 be an arbitrary real. As $L[G_\kappa] \models \varphi_m(x, y^\alpha, a_0^\alpha)$, we know that there is an a_2 such that for every a_3 and so on there is an a_{2n-2} such that $L[G_\kappa] \models \psi(x, y^\alpha, a_0^\alpha, a_1, ..., a_{2n-2})$. By case 1 in the definition of the iteration, this translates to $L[G_\kappa] \models$ "$(\#\psi, x, y^\beta, a_0^\beta, a_1, ..., a_{2n-2})$ is not coded". As a_1 was arbitrary we can conclude that

$$L[G_\kappa] \models \forall a_1 \exists a_2 ... \exists a_{2n-2}((\#\psi, x, y^\beta, a_0^\beta, a_1, .., a_{2n-2}) \text{ is not coded })$$

as desired.

$\square$

Lemma 4.2. *Again G_κ denotes a generic filter for the entire, κ-length iteration.*

1. *If α is least such that*

$$L[G_\kappa] \models \forall a_1 ... \exists a_{2n-2} \psi(x, y^\alpha, a_0^\alpha, ..., a_{2n-2}),$$

   then

$$L[G_\kappa] \models \exists a_1 \forall a_2 ... \forall a_{2n-2}((\#\psi, x, y^\alpha, a_0^\alpha, a_1, a_2, ..., a_{2n-2}) \text{ is coded.})$$

2. *If α is least such that*

$$L[G_\kappa] \models \forall a_1 ... \exists a_{2n-3} \psi(x, y^\alpha, a_0^\alpha, ..., a_{2n-3}),$$

   then

$$L[G_\kappa] \models \exists a_1 \forall a_2 ... \forall a_{2n-3}((\#\psi, x, y^\alpha, a_0^\alpha, a_1, a_2, ..., a_{2n-3}) \text{ is not coded.})$$

Proof. As α is the first ordinal for which

$$L[G_\kappa] \models \forall a_1 ... \exists a_{2n-2} \psi(x, y^\alpha, a_0^\alpha, ..., a_{2n-2}),$$

we know that $\forall \beta < \alpha$, $L[G_\kappa] \models \exists a_1 \forall a_2, ..., \forall a_{2n-2}(\neg\psi(x, y^\beta, a_0^\beta, a_1, ...))$. In particular, for every $\beta < \alpha$ there are reals a_1^β such that for every a_2^β there are reals a_3^β and so on such that $\neg\psi$ holds. Using the bijection π_α, we can find

19

an $a_1 = \pi_\alpha((a_1^\beta)_{\beta<\alpha})$ which has the property that for any real a_2 there will be a real $a_3 = \pi_\alpha(a_3^\beta)_{\beta<\alpha})$ such that for any real a_4 and so on $\neg\psi$ is true.

But this translates via the case 1 rule of our iteration to the assertion that

$$L[G_\kappa] \models \exists a_1 \forall a_2 ... \forall a_{2n-2}((\#\psi, x, y^\alpha, a_0^\alpha, a_1, a_2, ..., a_{2n-2}) \text{ is coded.})$$

$\square$

Lemma 4.3. *In $L[G_\kappa]$ the Σ_{n+1}^1-uniformization property holds true for every $n \geq 1$.*

Proof. Let $\varphi \equiv \exists a_0 \forall a_1 ... (\psi(x, y, a_0, a_1, ..., a_{2n-2}))$ be an arbitrary Σ_{2n+1}^1-formula in two free variables where ψ is Π_2^1. Let x be a real such that there is a real y with $L[G_\kappa] \models \varphi(x, y)$. We list all the triples $(x, y^\alpha, a_0^\alpha)$ according to our well-order $<$. Let α be least such that

$$L[G_\kappa] \models \forall a_1 ... (\psi(x, y^\alpha, a_0^\alpha, a_1, ...)).$$

Then by the last Lemma

$$L[G_\kappa] \models \exists a_1 \forall a_2 ... \forall a_{2n-2}((\#\psi, x, y^\alpha, a_0^\alpha, ...) \text{ is coded})$$

holds true. Note that this formula is Σ_{2n+1}^1. By the penultimate Lemma, for each $\beta > \alpha$

$$L[G_\kappa] \models \forall a_1 \exists a_2 ... \exists a_{2n-2}((\#\psi, x, y^\beta, a_0^\beta, a_1, ..) \text{ is not coded}),$$

which is Π_{2n+1}^1

So (x, y^α) is the unique pair satisfying the Σ_{2n+1}^1-formula

$$\exists a_0((\forall a_1 \exists a_2 ... \exists a_{2n-2}(\psi(x, y, a_0, ...) \wedge$$
$$\neg(\forall a_1 \exists a_2 ... \exists a_{2n-2}((\#\psi, x, y, a_0, a_1, ... a_{2n-2} \text{ is not coded}))$$

Indeed

$\square$

Lemma 4.4. $L[G_\kappa] \models \mathsf{BPFA}$.

Proof. This is standard. We show that $H(\omega_2)^{L[G_\kappa]}$ is Σ_1-elementary in any $L[G_\kappa]^\mathbb{P}$, where $\mathbb{P}$ is proper. We assume that the Σ_1-formula $\tau(x, p_0, ..., p_n)$ can be forced to become true with a proper $\mathbb{P} \in L[G_\kappa]$. The choice of F ensures that τ and its parameters are considered at an even stage $\beta < \kappa$ of our iteration. But $\mathbb{P}_\kappa * \mathbb{P}$ is suitable and witnesses that τ can be forced to become true. Hence we forced so that τ becomes true already at stage $\beta < \kappa$ and by upwards absoluteness of Σ_1-statements the proof is done. $\square$

References

[1] J. W. Addison *Some Consequences of the Axiom of Constructibilty.* Fundamenta Mathematicae 46, pp. 337-357, 1959.

[2] J. Bagaria, *Bounded forcing axioms as principles of generic absoluteness*, Archive for Mathematical Logic, vol. 39 (2000), no. 6, pp. 393–401.

[3] A. Caicedo and S.D. Friedman BPFA *and Projective Wellorderings of the Reals.* Journal of Symbolic Logic 76, pp. 1126-1136, (4) 2011.

[4] A. Caicedo and B. Velickovic *The bounded proper forcing axiom and wellorderings of the reals.* Mathematical Research Letters, 13 (2-3) (2006). pp. 393-408

[5] M. Foreman and M. Magidor *Mutually stationary sequences of sets and the non-saturation of the non-stationary ideal on $P_\kappa(\lambda)$.* Acta Math. 186 (2001), no. 2, 271–300.

[6] M. Goldstern and S. Shelah *The bounded proper forcing axiom.* The Journal of Symbolic Logic, Volume 60, Number 1. March 1995. pp.58–73.

[7] L. Harrington *Long projective wellorderings.* Ann. Math. Logic, 12(1):1–24, 1977.

[8] S. Hoffelner *Forcing the Π_3^1-Reduction Property and a Failure of Π_3^1-Uniformization*, Annals of Pure and Applied Logic 174 (8), 2023.

[9] S. Hoffelner *Forcing the Π_n^1-Uniformization Property*, submitted.

[10] S. Hoffelner *Forcing Axioms and the Uniformization Property*, submitted.

[11] S. Hoffelner *Forcing the Σ_3^1-Separation Property.* Journal of Mathematical Logic, 2022.

[12] S. Hoffelner, P. B. Larson, R. D. Schindler and L. Wu *Forcing Axioms and the Definability of the Nonstationary Ideal on the first Uncountable.* Journal of Symbolic Logic. Published online 2023:1-18. doi:10.1017/jsl.2023.40

[13] T. Jech *Set Theory. Third Millenium Edition.* Springer 2003.

[14] A. Kechris *Classical Descriptive Set Theory.* Springer 1995.

[15] R. D. Schindler *Set Theory. Exploring Independence and Truth* Springer 2014.

[16] S. Shelah *Proper and improper forcing.* Second edition. Perspectives in Mathematical Logic. Springer-Verlag, Berlin, 1998. xlviii+1020 pp

[17] J. Steel PFA *implies* $\mathsf{AD}^{L(\mathbb{R})}$. Journal of Symbolic Logic Vol. 70, No. 4, pp. 1255-1296, 2005.

[18] S. Todorcevic *A Note on the Proper Forcing Axiom,* Contemporary Mathematics vol. 31, 1984.

[19] S.Todorcevic *Localized reflection and fragments of PFA*, Set theory (Piscataway, NJ, 1999), volume 58 of DIMACS series in discrete mathematics and theoretical computer science, pp. 135–148. American Mathematics Society, Providence, RI (2002)